

AP Calculus AB Study Guide

Jake's Math Lessons

1 Review

1.1 A few important algebraic formulas

Dividing by a number is the same as multiplying by its reciprocal:

$$\frac{a}{b} = a * \frac{1}{b}$$

Examples:

$$\frac{3}{5} = 3 * \frac{1}{5}$$

$$\frac{x}{\frac{1}{4}} = x * \frac{4}{1} = x * 4 = 4x$$

Difference of Squares:

$$a^2 - b^2 = (a + b)(a - b)$$

Examples:

$$x^2 - 16 = (x + 4)(x - 4)$$

$$x - 16 = (\sqrt{x} + 4)(\sqrt{x} - 4)$$

Note: This is not the same as $(a - b)^2$ which is equal to $(a - b)(a - b) = a^2 - 2ab + b^2$.

Sum/Difference of Cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

1.2 Important Functions

Make sure you understand these functions well. Know their domain, range, any other important properties, and can graph them.

$$y = x^2$$

$$y = x^3$$

$$y = x$$

$$y = \sqrt{x}$$

$$y = \frac{1}{x}$$

$$y = |x|$$

$$y = \ln(x)$$

$$y = e^x$$

$$y = \lfloor x \rfloor$$

$$y = \sin(x)$$

$$y = \cos(x)$$

$$y = \tan(x)$$

$$y = \csc(x)$$

$$y = \sec(x)$$

$$y = \cot(x)$$

1.3 Function Properties

Even Function: $f(-x) = f(x)$

Odd Function: $f(-x) = -f(x)$

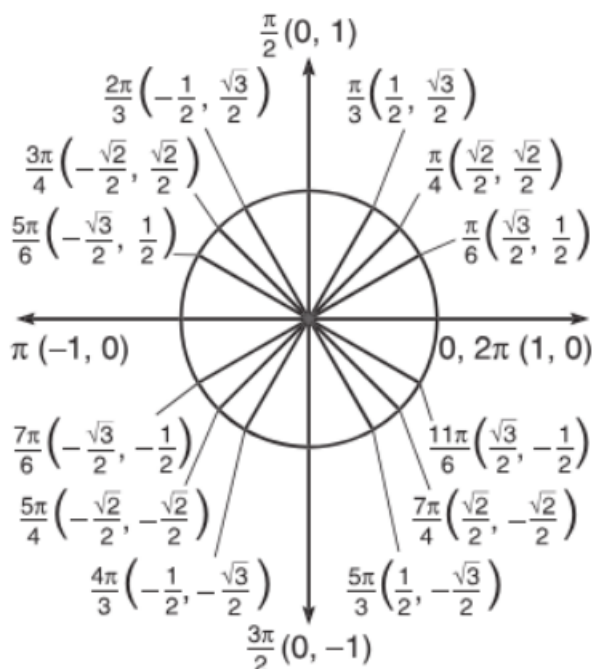
1.4 Inverse Functions

If $f^{-1}(x)$ is the inverse of $f(x)$, then

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

1.5 Trigonometry

Unit Circle:



From the unit circle, you can see for each angle:

$$\cos(\theta) = x$$

$$\sin(\theta) = y$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

Pythagorean Identities:

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Even and Odd Identities:

$$\sin(-x) = -\sin(x)$$

$$\csc(-x) = -\csc(x)$$

$$\tan(-x) = -\tan(x)$$

$$\cot(-x) = -\cot(x)$$

$$\cos(-x) = \cos(x)$$

$$\sec(-x) = \sec(x)$$

Double-angle Formulas:

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

Power-reducing Formulas:

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

Sum and Difference Formulas:

$$\sin(x \pm y) = \sin x \cos y \mp \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Note: The sign where the equations above say " $\pm$ " should be opposite of the sign where the equation says " $\mp$ ". For example: $\sin(x + y) = \sin x \cos y - \cos x \sin y$, or $\sin(x - y) = \sin x \cos y + \cos x \sin y$.

2 Limits

2.1 One-sided Limits

Left sided limit: $\lim_{x \rightarrow a^-} f(x) = b$ **Right sided limit:** $\lim_{x \rightarrow a^+} f(x) = b$

Existence of two-sided limits from one-sided limits

If $\lim_{x \rightarrow a^-} f(x) = b$ and $\lim_{x \rightarrow a^+} f(x) = b$, then $\lim_{x \rightarrow a} f(x) = b$.

If $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ DNE (Does Not Exist).

If $\lim_{x \rightarrow a^-} f(x)$ DNE, then $\lim_{x \rightarrow a} f(x)$ DNE.

If $\lim_{x \rightarrow a^+} f(x)$ DNE, then $\lim_{x \rightarrow a} f(x)$ DNE.

2.2 Limit Properties

In the below properties a , c , and n are all constants.

$$\lim_{x \rightarrow a} c = c$$

$$\lim_{x \rightarrow a} x = a$$

$$\lim_{x \rightarrow a} (cf(x)) = c \cdot \lim_{x \rightarrow a} f(x)$$

$$\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$$

$$\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$$

$$\lim_{x \rightarrow a} \left(\sqrt[n]{f(x)} \right) = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

2.3 Squeeze Theorem

If **1.** $g(x) \leq f(x) \leq h(x)$, **2.** $\lim_{x \rightarrow a} g(x) = L$, and **3.** $\lim_{x \rightarrow a} h(x) = L$

Then $\lim_{x \rightarrow a} f(x) = L$

2.4 Continuity

Definition: A function $f(x)$ is continuous at $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$.

Functions that are continuous everywhere:

Polynomials $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

Functions that are continuous on their entire domain:

Rational functions	$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_2 x^2 + b_1 x + b_0}$
Root functions	$y = \sqrt[n]{x}$
Trigonometric functions	$y = \sin(x), y = \cos(x), y = \tan(x), \dots$
Exponential functions	$y = e^x, y = a^x$
Logarithmic functions	$y = \log_a(x), y = \ln(x)$

2.5 L'Hospital's Rule

If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ OR if $\lim_{x \rightarrow a} f(x) = \pm\infty$ AND $\lim_{x \rightarrow a} g(x) = \pm\infty$

then, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

3 Derivatives

3.1 Limit Definition of a Derivative

Definition for a general derivative: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

Definition for the slope at a specific point: $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$.

3.2 Trig Functions

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\sin^2(x) + \cos^2(x) = 1$$

You can use the quotient rule to find the derivatives of some of these trig functions. First rewrite each in terms of just $\sin(x)$ and $\cos(x)$, then use other derivative rules from there.

$$\frac{d}{dx} \sin(x) = \cos(x)$$

$$\frac{d}{dx} \csc(x) = \frac{d}{dx} \frac{1}{\sin(x)} = -\frac{\cos(x)}{\sin^2(x)} = -\csc(x)\cot(x)$$

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

$$\frac{d}{dx} \sec(x) = \frac{d}{dx} \frac{1}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x)$$

$$\frac{d}{dx} \tan(x) = \frac{d}{dx} \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx} \cot(x) = \frac{d}{dx} \frac{\cos(x)}{\sin(x)} = -\frac{1}{\sin^2(x)} = -\csc^2(x)$$

$$\begin{aligned}\frac{d}{dx} \arcsin(x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \operatorname{arccsc}(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx} \arccos(x) &= -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \operatorname{arcsec}(x) &= \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx} \arctan(x) &= \frac{1}{1+x^2} & \frac{d}{dx} \operatorname{arccot}(x) &= -\frac{1}{1+x^2}\end{aligned}$$

3.3 Other Important Derivatives

In the below properties a and c are constants.

$$\begin{aligned}\frac{d}{dx} x &= 1 & \frac{d}{dt} c &= 0 \\ \frac{d}{dx} \ln(x) &= \frac{1}{x} & \frac{d}{dx} \log_a(x) &= \frac{1}{x \ln(a)} \\ \frac{d}{dx} e^x &= e^x & \frac{d}{dx} a^x &= \ln(a) \cdot a^x\end{aligned}$$

3.4 Important Rules

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Bring the power down in front and lower the power by 1.

Product Rule: $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

The derivative of the first times the second plus the derivative of the second times the first.

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$

Lo d Hi minus Hi d Lo, all over Lo squared.

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

The derivative of the outside, leave the inside function alone. Then multiply by the derivative of the inside.

Derivative of an Inverse: $\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$

3.5 Optimizing a Function

Local (Relative) Extrema

Critical values: Set $f'(x) = 0$ and solve for x . Also any x values where $f'(x)$ does not exist.

First Derivative Test: Plug x values around your critical values into $f'(x)$.

1. The critical point is a local max if $f'(x) > 0$ to the left and $f'(x) < 0$ to the right

of the critical point.

2. The critical point is a local min if $f'(x) < 0$ to the left and $f'(x) > 0$ to the right of the critical point.

3. The critical point is not a local max or a local min if $f'(x) > 0$ on both sides of the critical point or $f'(x) < 0$ on both sides.

Second Derivative Test: Plug your critical values into $f''(x)$.

1. The critical point is a local max if $f''(x) < 0$ at the critical point.

2. The critical point is a local min if $f''(x) > 0$ at the critical point.

3. The critical point could be a local max, a local min, or neither if $f''(x) = 0$ at the critical point.

Global (Absolute) Extrema

To find the global extrema on an interval $[a, b]$, plug all critical values within the interval, $x = a$, and $x = b$ into $f(x)$. You need to test the critical values and endpoints of the interval.

The absolute max will be whichever of these values that gives the *largest* output.

The absolute min will be whichever of these values that gives the *smallest* output.

3.6 Implicit Differentiation and Related Rates

4 steps to solve any related rates problem:

1. **Draw a sketch** - Just draw the situation described in the problem. It will make it much easier to visualize what's happening to create an equation.
2. **Come up with your equation** - Create an equation based on geometric formulas, like volumes or Pythagorean Theorem. This should only contain measurements like distances or dimensions of figures, NO RATES OF CHANGE YET.
3. **Implicit differentiation** - Take the derivative of both sides of your equation, usually with respect to time. This will introduce your rates of change.
4. **Solve for desired rate of change** - Isolate the rate of change the question asked you to find and plug in all the values for the other variables in the equation.

Example

A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s . At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

Using the fact that the *tangent* of an angle of a right triangle is equal to the opposite side divided by the adjacent side:

$$\tan(\theta) = \frac{100}{a} = 100a^{-1}$$

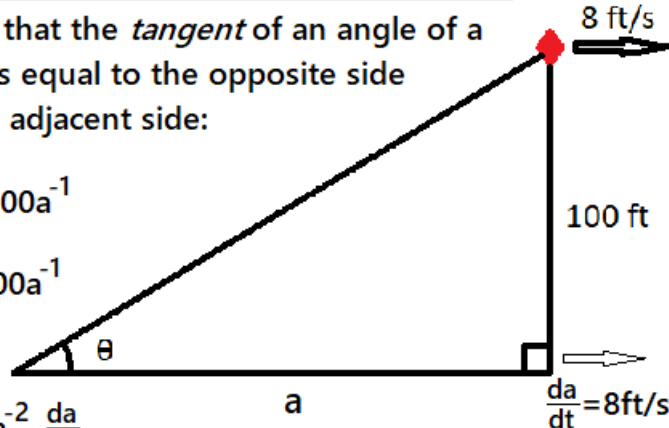
$$\frac{d}{dt} \tan(\theta) = \frac{d}{dt} 100a^{-1}$$

$$\frac{1}{\cos^2 \theta} \frac{d\theta}{dt} = -100a^{-2} \frac{da}{dt}$$

$$\frac{d\theta}{dt} = -100a^{-2} \frac{da}{dt} \cos^2 \theta$$

$$\frac{d\theta}{dt} = -100(100\sqrt{3})^{-2} \cdot 8 \cdot \cos^2 \frac{\pi}{6}$$

$$\frac{d\theta}{dt} = -\frac{1}{50}$$



$$\frac{da}{dt} = 8\text{ ft/s}$$

$$100^2 + a^2 = 200^2$$

$$a = 100\sqrt{3}$$

$$\tan(\theta) = \frac{100}{100\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

4 Integrals - Definition and Properties

4.1 Definition of a Definite Integral

If f is a function that's defined for $a \leq x \leq b$, and we split the interval $[a, b]$ into n pieces of equal width $\Delta x = \frac{b-a}{n}$. Then we say that $x_0, x_1, x_2, \dots, x_n$ are the endpoints of these smaller pieces where $x_0 = a$ and $x_n = b$. So $x_1^*, x_2^*, \dots, x_n^*$ would be sample points within each of the split pieces of the overall interval, so x_i^* is in the subinterval $[x_{i-1}, x_i]$. Then we can define the integral

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Remember, this represents the sum of the area of a bunch of infinitely thin rectangles. The height of each rectangle is the height of the function at each little piece ($f(x_i^*)$). And the width of each rectangle is the distance you go over to the side to get to the next rectangle (Δx).

4.2 Left Endpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_{i-1})\Delta x$$

$$\text{where } \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_{i-1} = a + (i-1)\Delta x$$

4.3 Right Endpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_i)\Delta x$$

$$\text{where } \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x$$

4.4 Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(\bar{x}_i)\Delta x = \Delta x [f(\bar{x}_1) + \dots + f(\bar{x}_n)]$$

$$\text{where } \Delta x = \frac{b-a}{n} \quad \text{and} \quad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

4.5 Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2n} [f(a) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(b)]$$

4.6 Definite Integral Properties

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b c dx = c(b-a), \text{ where } c \text{ is a constant}$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx, \text{ where } c \text{ is a constant}$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

$$\text{If } f(x) \geq 0 \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x)dx \geq 0$$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

4.7 Fundamental Theorem of Calculus

4.7.1 Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$, then the function g is defined by

$$g(x) = \int_a^x f(t) dt \quad a \leq x \leq b$$

is an antiderivative of f , meaning $g'(x) = f(x)$ for $a < x < b$.

4.7.2 Fundamental Theorem of Calculus Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f , meaning $F' = f$.

5 Integrals - Techniques for Evaluating

5.1 u-substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

When using u-substitution, you usually want to look for some piece of your function that has its derivative somewhere else in the function. Make that piece equal u . Then when you find du (the derivative of u), that will be somewhere in the function as well.

We can also use u-substitution for definite integrals. If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

All this means is that you have to convert your bounds of the integral to be in terms of u as well. Just plug the bounds of the original integral into the piece you have called u to convert them to be in terms of u .

5.2 Integrals of Symetric Functions

As long as f is continuous on $[-a, a]$.

1. If f is even [which means: $f(-x) = f(x)$], then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
2. If f is odd [which means: $f(-x) = -f(x)$], then $\int_{-a}^a f(x) dx = 0$.

5.3 Improper Integrals

5.3.1 Infinite Intervals

- (a) If $\int_a^t f(x) dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

assuming this limit exists.

- (b) If $\int_t^b f(x) dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided this limit exists as a finite number.

The improper integrals $\int_a^\infty f(x) dx$ and $\int_{-\infty}^b f(x) dx$ are **convergent** if the limit exists, and **divergent** if the limit does not exist.

- (c) If both $\int_a^\infty f(x) dx$ and $\int_{-\infty}^a f(x) dx$ are convergent, then for any real number a

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

- (d) $\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

5.4 Discontinuous Integrands

- (a) If f is continuous on $[a, b)$ and is discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

if this limit exists as a finite number.

- (b) If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

if this limit exists as a finite number.

The improper integral $\int_a^b f(x) dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c , where $a < c < b$, and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are convergent, then we define

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

6 Applications of Integration

The area A of the region bounded between the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, and $x = b$ where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$.

$$A = \int_a^b [f(x) - g(x)] dx$$

6.1 Solids of Revolution

Washer or Disk Method: $\pi \int_a^b (\text{outer radius})^2 - (\text{inner radius})^2 dx$

You'll need to figure out the inner and outer radius by considering the distance between the radii and the axis of rotation.

Cylindrical Shell Method: $\int 2\pi rh dx$

6.2 Average Value of a Function

The average value of a function f on the interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

7 Differential Equations

7.1 Separable Differential Equations

A separable differential equation is one where you can separate all of the x terms and dx on one side of the equation and all the y terms and dy on the other side. They will be in the form $\frac{dy}{dx} = g(x)f(y)$. They can be solved with the following steps:

$$\frac{dy}{dx} = g(x)f(y)$$

$$\frac{dy}{f(y)} = g(x) \, dx$$

$$\int \frac{1}{f(y)} \, dy = \int g(x) \, dx$$

Then integrate and solve for y .

Similarly, we can also have a differential equation in the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ and can be solved with a similar process:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$h(y) \, dy = g(x) \, dx$$

$$\int h(y) \, dy = \int g(x) \, dx$$

Then integrate and solve for y .

7.2 Exponential Growth and Decay

The solution to the initial-value problem for some constant k

$$\frac{dy}{dt} = ky \quad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt}$$

Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_s)$$

where $T(t)$ is the temperature of the object at time t , and T_s is the temperature of the surroundings.

Continuously Compounded Interest:

$$A(t) = A_0 e^{rt}$$

where $A(t)$ is the value of the investment at time t , A_0 is the value of the investment at time $t = 0$, and r is the annual interest rate of the continuous compound interest (as a decimal, not a percent).